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Additional Mathematics

附加數學科

Mathematical Induction

CE/AMT/NOTES/MI

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    數學歸納法的題形變化和應用

HKCEE Syllabus 會考課程

Mathematical Induction and its simple applications.

數學歸納法及其簡易應用

- **Application to proof of inequalities is <u>not</u> required.**

  <u>不包括</u>應用於不等式之證明

(A) 3 Standard steps for solving M.I. questions.

i. What is Mathematical Induction

Consider the following proposition P(n).

P(n): $1+2+3+...+n=\frac{1}{2}n(n+1)$ **for all natural numbers n.**

How to prove this pattern is true for all natural numbers n?

When $n=1$, **L.H.S.** $=1$

R.H.S. $=\frac{1}{2}(1)(1+1)=1$

When $n=2$, **L.H.S.** $=1+2=3$

R.H.S. $=\frac{1}{2}(2)(2+1)=3$

When $n=3$, **L.H.S.** $=1+2+3=6$

R.H.S. $=\frac{1}{2}(3)(3+1)=6$

When n = 4, 5, 6, …, 97, 98, 99,

:

When $n=100$, **L.H.S.** $=1+2+3+......+100=5050$

R.H.S. $=\frac{1}{2}(100)(100+1)=5050$

How about n = 1000, 10000, 100000,…??? Infinite natural numbers !!!
Therefore, …

Mathematical Induction 數學歸納法

When to Use : To prove a proposition P(n) is true for all natural numbers n.

Necessary Condition: P(n) can be proved if considering successive cases step by step.

ii. 3 Standard Steps for M.I. 數學歸納法三個基本步驟

Mathematical Induction 數學歸納法

Step 1:Prove P(0/1/2) is true
Step 2: Assume P(k) is true
Step 3: Consider P(k+1), Prove P(k+1) is true using P(k) is true assumption.

e.g.1 Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $1+2+3+......+n=\frac{1}{2}n(n+1)$ **for all natural numbers n.**

Sol: When $n=1$,

L.H.S. $=1$

R.H.S. $=\frac{1}{2}(1)(1+1)=1$

$\therefore$ L.H.S. = R.H.S., P(1) is true.

} **Step 1**

Assume P(k) is true, i.e. $1+2+3+......+k=\frac{1}{2}k(k+1)$ } **Step 2**

where k is a positive integer.

Consider P(k+1),

L.H.S. $=1+2+3+......+k+k+1$

$=\frac{1}{2}(k)(k+1)+(k+1)$ **[By Assumption P(k) is true]**

$=\frac{1}{2}(k+1)(k+2)$

} **Step 3**

$=\frac{1}{2}(k+1)(k+2)$

= R.H.S.

$\therefore$ P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n

p.s. Question Type: Simple Equality !!!

(B) Different Types of M.I. questions

Four Common Types of M.I. Questions

a. Simple Equality

b. Fraction Equality

c. Divisible

d. Derivation (Variety)

黎 Sir 提提你 :

Exam Type Questions:

a. Simple Equality 簡單等式

Skill 1: 無中生有大法

Skill 2: 望住最後一步

e.g.2 Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $1+2+3+......+n=\frac{1}{2}n(n+1)$ **for all natural numbers n.**

黎 Sir 提提你 :

Sol: When 當 $n=1$,

L.H.S. $=1$

R.H.S. $=\frac{1}{2}(1)(1+1)=1$ } **Step 1**

$\therefore$ **L.H.S. = R.H.S., P(1) is true. P(1) 成立.**

Assume 假設 P(k) is true, i.e. $1+2+3+......+k=\frac{1}{2}k(k+1)$

Where k is a positive integer. } **Step 2**

Consider 考慮 P(k+1),

L.H.S. $=1+2+3+......+k+k+1$

$=\frac{1}{2}(k)(k+1)+(k+1)$ **[By Assumption P(k) is true]**

$=\frac{1}{2}(k+1)(k+2)$ 無中生有大法!!!

$=\frac{1}{2}(k+1)(k+2)$ 望住最後一步!!!

$=$ **R.H.S.** } **Step 3**

$\therefore$ **P(k+1) is true, <u>if P(k) is true.</u>**

By M.I., P(n) is true for all natural numbers n

e.g.3 Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}$ **for all natural numbers n.**

黎 Sir 提提你 :

When n=1,

L.H.S. $= 1^2 = (-1)^{1-1} \dfrac{1(1+1)}{2} =$ **R.H.S.**

So P(1) is true.

Assume P(k) is true, i.e. $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{k-1} k^2 = (-1)^{k-1} \dfrac{k(k+1)}{2}$,

where k is a positive integer.

Consider P(k+1),

L.H.S. $= 1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{k-1} k^2 + (-1)^{k+1-1} (k+1)^2$

$= (-1)^{k-1} \dfrac{k(k+1)}{2} + (-1)^{k+1-1} (k+1)^2$ **[By Assumption P(k) is true]**

$= (-1)^k \dfrac{(k+1)}{2} [(-1)^{-1} k + 2(k+1)]$ 無中生有大法!!!

$= (-1)^k \dfrac{(k+1)}{2} [(-1)k + 2k + 2]$ **(-1)^k = -1 if k is odds !!!**
(-1)^k = 1 if k is even !!!

$= (-1)^k \dfrac{(k+1)}{2} [(-1k + 2k + 2]$

$= (-1)^k \dfrac{(k+1)}{2} [(k+2]$

$= (-1)^k \dfrac{(k+1)(k+2)}{2}$

$= (-1)^{k+1-1} \dfrac{(k+1)(k+2)}{2}$ 望住最後一步!!!

= R.H.S.

∴ P(k+1) is true, <u>if P(k) is true.</u>

By M.I., P(n) is true for all natural numbers n

b. Fraction Equality 份數等式

Skill 1: 無中生有大法 + / 通份母

Skill 2: 望住最後一步

e.g.4 Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+...+\frac{n\times 2^n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$ **for all natural numbers n.**

黎 Sir 提提你:

When n=1,

L.H.S. $=\frac{1\times 2}{2\times 3}=\frac{1}{3}$

R.H.S. $=\frac{2^{n+1}}{n+2}-1=\frac{1}{3}=$ **L.H.S.**

So P(1) is true.

Assume P(k) is true, i.e. $\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+...+\frac{k\times 2^n}{(k+1)(k+2)}=\frac{2^{k+1}}{k+2}-1$,

where k is a positive integer.

Consider P(k+1),

L.H.S. $=\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+...+\frac{k\times 2^k}{(k+1)(k+2)}+\frac{(k+1)\times 2^{k+1}}{(k+1+1)(k+1+2)}$

$=\frac{2^{k+1}}{k+2}-1+\frac{(k+1)\times 2^{k+1}}{(k+2)(k+3)}$ **[By Assumption P(k) is true]**

$=\frac{2^{k+1}}{k+2}(1+\frac{k+1}{k+3})-1$ 無中生有大法!!!

$=\frac{2^{k+1}}{k+2}(\frac{2k+4}{k+3})-1$ 通份母!!!

$=\frac{2^{k+1}}{k+2}\bullet\frac{2(k+2)}{k+3}-1$

$=\frac{2^{k+2}}{k+3}-1$

$=\frac{2^{(k+1)+1}}{(k+1)+2}-1$ 望住最後一步!!!

= R.H.S.

∴ P(k+1) is true, if P(k) is true.

By M.I., P(n) is true for all natural numbers n

e.g.5 Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$ **for all natural numbers n.**

黎 Sir 提提你 :

When n=1,

L.H.S. $=\frac{1}{2^1}=\frac{1}{2}$

R.H.S. $=2-\frac{1+2}{2^1}=\frac{1}{2}$ **L.H.S.**

So P(1) is true.

Assume P(k) is true, i.e. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{k}{2^k}=2-\frac{k+2}{2^k}$**, where k is a positive integer.**

Consider P(k+1),

L.H.S. $=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}}$

$=2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$ **[By Assumption P(k) is true]**

$=2+\frac{-2(k+2)+(k+1)}{2^{k+1}}$ 通份母!!!

$=2+\frac{-2k-4+k+1}{2^{k+1}}$

$=2+\frac{-k-3}{2^{k+1}}$

$=2+\frac{-(k+3)}{2^{k+1}}$

$=2-\frac{k+3}{2^{k+1}}$

$=2-\frac{(k+1)+2}{2^{k+1}}$ 望住最後一步!!!

= R.H.S.

$\therefore$ **P(k+1) is true, <u>if P(k) is true.</u>**

By M.I., P(n) is true for all natural numbers n

c. Divisible 整除性

Skill 1: Let 設 P(n) be $a = bn$

Skill 2: $x^{k+1} \Rightarrow x \bullet x^k$

Skill 3: 數字分拆

Skill 4: 抽公因式(Common Factor)

Skill 5: 望住最後一步

e.g.6 Prove, by M.I., $8^n - 3^n$ **is divisible by 5 for all positive integer n.**

黎 Sir 提提你 :

Let P(n) be the statement: $8^n - 3^n = 5M$ **where M is a integer**

When $n = 1$**,**

L.H.S. $= 8^1 - 3^1 = 5 = 5(1) =$ **R.H.S.** → Skill 1 - Let 設 P(n) be $a = bn$

∴ **P(1) is true.**

Assume 假設 P(k) is true, i.e. $8^k - 3^k = 5Q$ **where Q is a integer.**

Consider P(k+1),

L.H.S. $= 8^{k+1} - 3^{k+1}$

$= 8 \bullet 8^k - 3 \bullet 3^k$ → Skill 2 - $x^{k+1} \Rightarrow x \bullet x^k$

$= (5+3) \bullet 8^k - 3 \bullet 3^k$ → Skill 3 數字分拆

$= 5 \bullet 8^k + 3 \bullet 8^k - 3 \bullet 3^k$

$= 5 \bullet 8^k + 3 \bullet (8^k - 3^k)$ → Skill 4 - Common Factor 抽公因式

$= 5 \bullet 8^k + 3 \bullet (5Q)$ → [By Assumption P(k) is true]

$= 5 \bullet (8^k + 3Q)$

$= 5 \bullet (R)$ **where** $R = 8^k + 3Q$ **is an integer**

$= 5 \bullet (R)$

= R.H.S. → Skill 5 -望住最後一步

∴ **P(k+1) is true, <u>if P(k) is true.</u>**

By M.I., P(n) is true for all natural numbers n

e.g.7 Prove, by M.I., $9^n - 1$ is divisible by 8 for all positive integer n.

黎 Sir 提提你 :

Let P(n) be the statement: $9^n - 1 = 8M$ **where M is a positive integer**

When $n = 1$,

L.H.S. $= 9^1 - 1 = 8(1) =$ **R.H.S.**

Skill 1 - Let P(n) be $a = bn$

$\therefore$ **P(1) is true.**

Assume P(k) is true, i.e. $9^k - 1 = 8Q$ **where Q is a positive integer.**

Consider P(k+1),

L.H.S. $= 9^{k+1} - 1$

$= 9^k \bullet 9 - 1$ — **Skill 2 -** $x^{k+1} \Rightarrow x \bullet x^k$

$= 9^k \bullet (8+1) - 1$ — **Skill 3** 數字分拆

$= 9^k \bullet 8 + 9^k - 1$

$= 9^k \bullet 8 + 8Q$ **[By Assumption P(k) is true]**

$= 8(9^k + Q)$ — **Skill 4 - Common Factor** 抽公因式

$= 8 \bullet R$ **where** R **is an integer**

= R.H.S. — **Skill 5 -** 望住最後一步

$\therefore$ **P(k+1) is true, <u>if P(k) is true.</u>**

By M.I., P(n) is true for all natural numbers n

d. Derivation (Variety) 引伸 (變化)

Skill 1: 數字分拆

Skill 2: 併項法

Skill 3: 斬頭切尾

e.g.8 a. Prove, by M.I., P(n) is true for all natural numbers n.

P(n): $1\times2+2\times3+3\times4+...+n(n+1)=\frac{1}{3}n(n+1)(n+2)$ **for all natural numbers n.**

b. Hence , evaluate $1\times3+2\times4+3\times5+...+50(52)$ **.**

黎 Sir 提提你 :

When n=1,

L.H.S. $=1\times(1+1)=1\times2=2=\frac{1}{3}(1)(2)(3)=$ **R.H.S.**

$\therefore$ **P(1) is true.**

Assume P(k) is true, i.e. $1\times2+2\times3+3\times4+...+k(k+1)=\frac{1}{3}k(k+1)(k+2)$**,**

where k is a positive integer.

Consider P(k+1),

L.H.S. $=1\times2+2\times3+3\times4+...+k(k+1)+(k+1)(k+1+1)$

$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+1+1)$ **[By Assumption P(k) is true]**

$=\frac{1}{3}(k+1)(k+2)(k+3)$ 無中生有大法!!!

$=\frac{1}{3}(k+1)(k+1+1)(k+1+2)$ 望住最後一步!!!

= R.H.S.

$\therefore$ **P(k+1) is true, if P(k) is true.**

By M.I., P(n) is true for all natural numbers n

b. $1\times3+2\times4+3\times5+...+50(52)$

$=1\times(1+2)+2\times(1+3)+3\times(1+4)+...+50(1+51)$ **Skill 1** 數字分拆

$=1\times1+1\times2+2\times1+2\times3+3\times1+3\times4+...+50\times1+50\times51$

$=(1\times1+2\times1+3\times1+...+50\times1)+(1\times2+2\times3+3\times4+...+50\times51)$ **Skill 2** 併項法

$=\dfrac{(1+50)(50)}{2}+\dfrac{1}{3}(50)(51)(52)$ (By part a)

$=45475$ **Sum of A.P. !!!**

e.g.9 (a) Prove, by mathematical induction, that
$2(2)+3(2^2)+4(2^3)+......+(n+1)(2^n) = n(2^{n+1})$ **for all positive integers *n*.**

(b) Show that $1(2)+2(2^2)+3(2^3)+......+98(2^{98})=97(2^{99})+2$.

黎 Sir 提提你 :

(a) Let P(n) be the statement $2(2)+3(2^2)+4(2^3)+......+(n+1)(2^n) = n(2^{n+1})$
for all positive integers n.
When n=1, L.H.S. $=2(2)=2(2^{1+1})=$ **R.H.S.**
So P(1) is true.

Assume P(k) is true, i.e. $2(2)+3(2^2)+4(2^3)+......+(k+1)(2^k)=k(2^{k+1})$

Consider P(k+1),

L.H.S. $=2(2)+3(2^2)+4(2^3)+......+(k+1)(2^k)+(k+1+1)(2^{k+1})$

$=k(2^{k+1})+(k+2)(2^{k+1})$ **[By Assumption P(k) is true]**

$=(k+1)[\frac{k}{k+1}(2^{k+1})+\frac{k+2}{k+1}(2^{k+1})]$ 無中生有大法!!!

$=(k+1)[\frac{k(2^{k+1})+(k+2)(2^{k+1})}{k+1}]$

$=(k+1)[\frac{2(k+1)(2^{k+1})}{k+1}]$

$=(k+1)[2(2^{k+1}]$

$=(k+1)(2^{k+2)}$

$=(k+1)(2^{k+1+1)}$ **= R.H.S.** 望住最後一步!!!

$\therefore$ **P(k+1) is true, <u>if P(k) is true.</u>**

By M.I., P(n) is true for all natural numbers n

(b) $1(2)+2(2^2)+3(2^3)+......+98(2^{98})$

$=(2-1)(2)+(3-1)(2^2)+(4-1)(2^3)+......+(99-1)(2^{98})$ → **Skill 1: 數字分拆**

$=(2)(2)+(3)(2^2)+(4)(2^3)+......+(99)(2^{98})-[(1)(2)+(1)(2^2)+(1)(2^3)+......+(1)(2^{9}$

$=98(2^{98+1})-[(2)+(2^2)+(2^3)+......+(2^{98})]$ **Skill 2 併項法(Grouping Terms)**

$=98(2^{99})-[\frac{2(1-2^{98})}{1-2}]$ → **Sum of G.P. !!!**

$=97(2^{99})+2^{99}+[\frac{2-2^{99}}{1}]$

$=97(2^{99})+2$

e.g.10 (a) Prove, by mathematical induction, that

$$\frac{1}{1\bullet 2\bullet 3\bullet 4}+\frac{1}{2\bullet 3\bullet 4\bullet 5}+\frac{1}{3\bullet 4\bullet 5\bullet 6}+\ldots\ldots+\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$$

for all positive integers *n*.

(b) Evaluate

$$\frac{1}{9\bullet 10\bullet 11\bullet 12}+\frac{1}{10\bullet 11\bullet 12\bullet 13}+\frac{1}{11\bullet 12\bullet 13\bullet 14}+\ldots\ldots+\frac{1}{19(20)(21)(22)}.$$

黎 Sir 提提你 :

(a)

(b)

e.g.11 (a) Prove, by mathematical induction, that

$$1^2+2^2+3^2+\ldots\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$$ **for all positive integers *n*.**

(b) **Evaluate** $2^2+4^2+6^2+\ldots\ldots+100^2$.

(c) **Evaluate** $141\times142+142\times143+143\times144+\ldots\ldots+180\times181$.

黎 Sir 提提你 :

(a)

(b)

(c)

e.g.12 a) Prove, by M.I., $32^{2n-1}+1$ is divisible by 11 for all positive integer n.

利用數學歸納法, 對於所有正整數 ***n***,, 證明 $32^{2n-1}+1$ 能被 **11** 整除.

b) Is $2^{985}+1$ divisible by 11? Please support you answers with reasons.

黎 Sir 提提你 :

(a)

(b)

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