

# Mathematics

# 數學科

## Binomial Theorem

## 二項式定理



計數要小心，  
咪期望快一陣！

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### HKCEE Syllabus 會考課程

#### **The Binomial Theorem for positive integral indices**

正整指數之二項式定理

- **Excluding determination of the greatest term and relations between coefficients.**

不包括係數間之關係及求最大項

## (A) Definition of factorial relationship with nCr 階乘的定義和與 nCr 的關係

### i. Definition of Factorial 階乘的定義:

$n! = n(n-1)(n-2)\dots(3)(2)(1)$  where  $n$  is a positive integer 正整數.

$0! = 1$  [Definition of Zero Factorial 零階乘的定義]

e.g.

$$10! = (10)(9)(8)\dots(3)(2)(1)$$

e.g.

$$(n-1)! = (n-1)(n-2)(n-3)\dots(3)(2)(1)$$

e.g.

$$\frac{9!}{5!} = \frac{9 \times 8 \times 7 \times \dots \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

e.g.

$$\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)\dots(3)(2)(1)}{(n-3)(n-2)(n-1)\dots(3)(2)(1)}$$

$$= n(n-1)(n-2)$$

$$= 9 \times 8 \times 7 \times 6$$

$$= 3024$$

### ii. Definition of nCr nCr 的定義:

$$nCr = \frac{n!}{(n-r)!r!} \text{ where } n \text{ and } r \text{ is positive and } n \geq r.$$

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### iii. Common Properties of $nCr$ :

1.  $C_r^n = C_{n-r}^n$
2.  $C_0^n = C_n^n = 1$
3.  $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$  or  $C_{r-1}^n + C_r^n = C_r^{n+1}$

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e.g. Prove  $C_r^n = C_{n-r}^n$ .

Sol: L.H.S. =  $C_r^n$

$$= \frac{n!}{(n-r)!r!}$$

$$= \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{[n-(n-r)]!(n-r)!}$$

$$= C_{n-r}^n = R.H.S.$$

So  $C_r^n = C_{n-r}^n$

e.g. Prove  $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$ .

Sol: L.H.S. =  $C_r^n + C_{r+1}^n$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{[n-(r+1)]!(r+1)!}$$

$$= \frac{n!(r+1) + n!(n-r)}{(n-r)!(r+1)!} = \frac{n! \bullet r + n! + n \bullet n! - n! \bullet r}{(n-r)!(r+1)!}$$

$$= \frac{n \bullet n! + n!}{(n+1-r-1)!(r+1)!} = \frac{n!(n+1)}{(n+1-r-1)!(r+1)!}$$

$$= \frac{(n+1)!}{[n+1-(r+1)]!(r+1)!} = C_{r+1}^{n+1} = R.H.S..$$

So  $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$

做緊咩?



#### iv. Pascal's Triangle 楊輝三角

nCr when	n=0						1
	n=1					1	1
	n=2				1	2	1
	n=3			1	3	3	1
	n=4	1	4	6	4	1	
	.....	$C_0^4$	$C_1^4$	$C_2^4$	$C_3^4$	$C_4^4$	
							.....

### (B) Explanation of Binomial Theorem

#### 二項式定理詳解

i. Consider Expanding 考慮展開  $(a+b)^n = (a+b)(a+b)(a+b)\dots(a+b)$

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ii. Binomial Theorem 二項式定理

$$(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \dots + C_r^n a^{n-r}b^r + \dots + C_{n-1}^n ab^{n-1} + C_n^n b^n$$

$$(1+x)^n = 1 + C_1^n x + C_2^n x^2 + \dots + C_r^n x^r + \dots + C_{n-1}^n x^{n-1} + x^n \quad \text{When } a=1 \text{ and } b=x.$$

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**e.g. Expand.**  $(4m - 3y)^3$

**Sol:**  $(4m - 3y)^3$

$$\begin{aligned} &= C_0^3(4m)^3(-3y)^0 + C_1^3(4m)^2(-3y)^1 + C_2^3(4m)^1(-3y)^2 + C_3^3(4m)^0(-3y)^3 \\ &= 1(64m^3)(1) + 3(16m^2)(-3y) + 3(4m)(9y^2) + 1(1)(-27y^3) \\ &= 64m^3 - 144m^2y + 108my^2 - 27y^3 \end{aligned}$$

**e.g. Expand**  $\left(x + \frac{1}{x}\right)^4$

**Sol:**  $\left(x + \frac{1}{x}\right)^4$

$$\begin{aligned} &= C_0^4(x)^4\left(\frac{1}{x}\right)^0 + C_1^4(x)^3\left(\frac{1}{x}\right)^1 + C_2^4(x)^2\left(\frac{1}{x}\right)^2 + C_3^4(x)^1\left(\frac{1}{x}\right)^3 + C_4^4(x)^0\left(\frac{1}{x}\right)^4 \\ &= 1(x)^4(1) + 4(x)^3\left(\frac{1}{x}\right) + 6(x)^2\left(\frac{1}{x^2}\right) + 4(x)^1\left(\frac{1}{x^3}\right) + 1(x)^0\left(\frac{1}{x^4}\right) \\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \end{aligned}$$

**e.g. Expand**  $(1 + 2x)^3$

**Sol:**  $(1 + 2x)^3$

$$\begin{aligned} &= C_0^3(2x)^0 + C_1^3(2x)^1 + C_2^3(2x)^2 + C_3^3(2x)^3 \\ &= 1 + 3(2x) + 3(4x^2) + 1(8x^3) \\ &= 1 + 6x + 12x^2 + 8x^3 \end{aligned}$$

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## (C) General Term of Binomial Function

### 二項式函數的通項

#### i. General Term 通項

$$\text{General Term 通項} = C_r^n a^{n-r} b^r$$

$$(a+b)^n = a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \dots + C_r^n a^{n-r} b^r + \dots + C_{n-1}^n a b^{n-1} + b^n$$

1<sup>st</sup> Term    2<sup>nd</sup> Term    3<sup>rd</sup> Term                      r+1<sup>th</sup> Term                      nth Term                      n+1th Term

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e.g. Find the General Term 通項 of  $(1+2x)^3$

$$\text{Sol: General Term 通項} = C_r^n a^{n-r} b^r = C_r^3 1^{3-r} (2x) = C_r^3 2^r x^r$$

e.g. Find the General Term 通項 of  $\left(x + \frac{1}{x}\right)^4$

$$\text{Sol: General Term 通項} = C_r^n a^{n-r} b^r = C_r^4 (x)^{4-r} \left(\frac{1}{x}\right)^r = C_r^4 (x)^{4-r-r} = C_r^4 (x)^{4-2r}$$

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## (D) Variation of Binomial Theorem Questions 二項式定理問題的變化

### Exam Type Questions:

1. Direct Expansion 直接展開
2. Finding General Term 找通項
3. Finding Coefficients 找係數
4. Variation of Summation of  $nCr$   $nCr$  之和及變化
5. Finding  $n$  找  $n$

### 1. Direct Expansion 直接展開

#### Skill 1: Binomial Theorem 技巧 1: 二項式定理

e.g. Expand.  $(4m-3y)^3$       e.g. Expand  $\left(x+\frac{1}{x}\right)^4$       e.g. Expand  $(1+2x)^3$

Sol: Pls refer to p.5

### 2. Finding General Term 找通項

#### Skill 1: General Term (r+1th Term) $= C_r^n a^{n-r} b^r$

#### 技巧 1: 通項 (r+1th 項) $= C_r^n a^{n-r} b^r$

e.g. Find the General Term of  $(1+2x)^3$       e.g. Find the General Term of  $\left(x+\frac{1}{x}\right)^4$

Sol: Pls refer to p.6

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### 3. Finding Coefficients 找係數

#### Skill 1: 技巧 1: General Term 通項 = $C_r^n a^{n-r} b^r$

e.g. Find the Coefficient of  $x^3$  in the expansion of  $(2+3x)^5$

$$\begin{aligned}\text{Sol: when } r=3, \text{ The General Term} &= C_3^5 2^{5-3} (3x)^3 \\ &= 10(4)(27x^3) \\ &= 108x^3\end{aligned}$$

So the Coefficient of  $x^3$  is 108.

e.g. Find the Coefficient of  $x^2$  in the expansion of  $\left(x - \frac{2}{x}\right)^4$

$$\begin{aligned}\text{Sol: General Term of } \left(x - \frac{2}{x}\right)^4 &= C_r^4 x^{4-r} \left(-\frac{2}{x}\right)^r \\ &= C_r^4 x^{4-r-r} (-2)^r \\ &= C_r^4 (-2)^r x^{4-2r}\end{aligned}$$

When  $4-2r=2$ ,  $r=1$ .

So the coefficient of  $x^2$  is  $= C_1^4 (-2)^1 = 4(-2) = -8$

e.g. Determine and find (If any) whether the expansion of  $\left(2x^2 - \frac{1}{x}\right)^9$  consist of constant term and  $x^2$  term.

$$\text{Sol: General Term of } \left(2x^2 - \frac{1}{x}\right)^9 = C_r^9 (2x^2)^{9-r} \left(-\frac{1}{x}\right)^r = \frac{C_r^9 2^{9-r} x^{18-3r}}{(-1)^r} = \frac{C_r^9 2^{9-r} x^{18-3r}}{(-1)^r}$$

When  $18-3r=0$ ,  $r=6$ , So the Constant Term is  $= \frac{C_6^9 2^{9-6}}{(-1)^6} = 84 \cdot 8 = 672$

When  $18-3r=2$ ,  $r = \frac{16}{3} \neq \text{integer}$ , So there is no  $x^2$  term.

## Skill 2: Product of Binomial 技巧 2: 二項式相乘

e.g. Given  $(1+3x)^4(1-2x)^5 = 1+ax+bx^2+\dots$ , find the values of the constant  $a$  and the coefficient of  $x^2$   $b$ .

**Sol:** Given  $(1+3x)^4(1-2x)^5 = 1+ax+bx^2+\dots$

$$[1+c_1^4(3x)+c_2^4(3x)^2+\dots][1+C_1^5(-2x)+C_2^5(-2x)^2+\dots]=1+ax+bx^2+\dots$$

$$1+(3\cdot c_1^4-2\cdot C_1^5)x+(9\cdot C_2^4+4\cdot C_2^5+3\cdot C_1^4\cdot(-2)\cdot C_1^5)x^2+\dots=1+ax+bx^2+\dots$$

**By comparing Coefficients,**  $a=3\cdot c_1^4-2\cdot C_1^5$ ,  $b=9\cdot C_2^4+4\cdot C_2^5-6\cdot C_1^4\cdot C_1^5$

**So**  $a=3\cdot 4-2\cdot 5=2$ ,  $b=9\cdot 6+4\cdot 10-6\cdot 4\cdot 5=-26$

**So**  $a=2$ ,  $b=-26$

## Skill 3: Sum of Binomial 技巧 3: 二項式相加

e.g. Given  $(1+x)^6+(1+2x)^6$ , find the coefficient of  $x^2$

**Sol:** The coefficient of  $x^2 = C_2^6 + C_2^6(2)^2 = 15 + 15\cdot 4 = 75$

## Skill 4: 技巧 4: Trinomial 三項式

e.g. Given  $(1+x+ax^2)^6 = 1+6x+k_1x^2+k_2x^3+\dots$ , Express  $k_1$  and  $k_2$  in terms of  $a$ .

**Sol:**  $(1+x+ax^2)^6 = 1+6x+k_1x^2+k_2x^3+\dots$

$$1+C_1^6x(1+ax)+C_2^6x^2(1+ax)^2+C_3^6x^3(1+ax)^3+\dots=1+6x+k_1x^2+k_2x^3+\dots$$

$$1+C_1^6x+C_1^6ax^2+C_2^6x^2+2aC_2^6x^3+a^2C_2^6x^4+C_3^6x^3+\dots=1+6x+k_1x^2+k_2x^3+\dots$$

$$k_1 = C_1^6 + C_2^6 = 6 + 15 = 21, \quad k_2 = 2aC_2^6 + C_3^6 = 2\cdot 15 + 20 = 50$$

**So**  $k_1 = 21, k_2 = 50$

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## 4. Variation of Summation of $nCr$ $nCr$ 之和及變化

### Skill 1: Substitute $x=a$ 技巧 1: 代 $x=a$

e.g. By Considering and expanding  $(1+x)^5$ , find  $C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5$ .

Sol:  $(1+x)^5 = C_0^5 x^0 + C_1^5 x^1 + C_2^5 x^2 + C_3^5 x^3 + C_4^5 x^4 + C_5^5 x^5$

Put 代入  $x=1$ ,  $(1+1)^5 = C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5$   
 $C_0^5 + C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5$

### Skill 2: Differentiate before Substitute $x=a$

### 技巧 2: 取導數後再代 $x=a$

e.g. By Considering and expanding  $(1+x)^5$ , find  $C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5$ .

Sol:  $(1+x)^5 = C_0^5 x^0 + C_1^5 x^1 + C_2^5 x^2 + C_3^5 x^3 + C_4^5 x^4 + C_5^5 x^5$

Differentiate both side with respect to  $x$  在兩邊同時對  $x$  取導數

$$5(1+x)^4 = C_1^5 + 2C_2^5 x + 3C_3^5 x^2 + 4C_4^5 x^3 + 5C_5^5 x^4$$

Put 代入  $x=1$ ,  $5(1+1)^4 = C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5$   
 $C_1^5 + 2C_2^5 + 3C_3^5 + 4C_4^5 + 5C_5^5 = 80$

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## 5. Finding n 找 n

### Skill 1: 技巧 1: General Term 通項 = $C_r^n a^{n-r} b^r$

e.g. By expanding of  $(ax + \frac{1}{x^2})^n$  in descending powers of x where  $a > 0$ , if the 4<sup>th</sup> term of the expansion is independent of x and equals  $\frac{21}{2}$ , find the value of a and n.

$$\text{Sol: General Term} = C_r^n (ax)^{n-r} \left(\frac{1}{x^2}\right)^r = C_r^n a^{n-r} x^{n-r-2r} = C_r^n a^{n-r} x^{n-3r}$$

$$\text{The 4th Term} = C_3^n a^{n-3} x^{n-3(3)} = C_3^n a^{n-3} x^{n-9} = \frac{21}{2}$$

$$\text{So } n-9=0 \Rightarrow n=9 \text{ and So } C_3^9 a^{9-3} = \frac{21}{2} \Rightarrow 84a^6 = \frac{21}{2} \Rightarrow a^6 = \frac{21}{2 \cdot 84}$$

$$\Rightarrow a^6 = \frac{1}{8} \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}} \text{ or } a = -\frac{1}{\sqrt{2}} \text{ (Rejected)}$$

### Skill 2: 技巧 2: Sum of Binomials 二項式相加

e.g. Given  $(1+x)^n + (1+2x)^n$  and the coefficient of  $x^2$  is 75. Find n.

$$\text{Sol: The coefficient of } x^2 = C_2^n + C_2^n (2)^2 = 5 \cdot C_2^n = 75$$

$$C_2^n = 15$$

$$\frac{n!}{(n-2)!2!} = 15 \Rightarrow n(n-1) = 15 \cdot 2 \Rightarrow n^2 - n - 30 = 0$$

$$(n-6)(n+5) = 0 \Rightarrow n = 6 \text{ or } n = -5 \text{ (Rejected)}$$

$$\text{So } n = 6$$

### Skill 3: Fraction Equality: Indices Law $x^a \cdot x^b = x^{a+b}$

#### 技巧 3: 份數等式: 指數 $x^a \cdot x^b = x^{a+b}$

e.g. Given  $(x - \frac{3}{x})^2 (1+2x)^n$  and the constant term is 210. Find n.

$$\text{Sol: Expand } (x - \frac{3}{x})^2 (1+2x)^n \Rightarrow [x^2 - 2x(\frac{3}{x}) + (\frac{3}{x})^2][1 + C_1^n (2x) + C_2^n (2x)^2 + \dots]$$

$$\text{The constant term is: } -6 + 3^2 \cdot C_2^n \cdot 2^2 = 210 \Rightarrow -6 + 36 \cdot \frac{(n)(n-1)}{2} = 210$$

$$\Rightarrow -6 + 18n^2 - 18n = 210 \Rightarrow 18n^2 - 18n - 216 = 0 \Rightarrow n^2 - n - 12 = 0$$

$$\Rightarrow (n-4)(n+3) = 0 \Rightarrow n = 4 \text{ or } n = -3 \text{ (Rejected)}$$

$$\text{So } n = 4$$

## Skill 4: 技巧 4: Sum of Coefficients 係數運算

e.g. In the expansion of  $(1+x)^n$ , if the coefficient of  $x$ ,  $x^2$  and  $x^3$  are in Arithmetic Sequence 等比數列, find the value of  $n$ .

Sol:  $(1+x)^n = C_0^n x^0 + C_1^n x^1 + C_2^n x^2 + C_3^n x^3 + \dots$

Coefficient of  $x$ ,  $x^2$  and  $x^3$  are  $C_1^n$ ,  $C_2^n$  and  $C_3^n$  respectively

$$C_2^n = \frac{C_1^n + C_3^n}{2} \Rightarrow$$

$$\frac{n!}{(n-2)!(2!)} = \frac{1}{2} \left[ n + \frac{n!}{(n-3)!(3!)} \right] \Rightarrow \frac{n(n-1)}{(2)} = \frac{1}{2} \left[ n + \frac{n(n-1)(n-2)}{(6)} \right]$$

$$\Rightarrow n(n-1) = \left[ \frac{6n + n(n^2 - 3n + 2)}{(6)} \right] \Rightarrow 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$$

$$\Rightarrow n^3 - 9n^2 + 14n = 0 \Rightarrow n(n^2 - 9n + 14) = 0 \Rightarrow n(n-7)(n-2) = 0$$

$$\Rightarrow n = 0 \text{ (Rejected) or } n = 2 \text{ (Rejected) or } n = 7$$

$$\Rightarrow n = 7$$

## Skill 5: 技巧 5: Trinomial 三項式

e.g. Given the coefficient of  $x^2$  after the expansion of  $(1+x+x^2)^n$  is 21. Find  $n$ .

Sol:  $(1+x+x^2)^n = [1+x(1+x)]^n = 1 + C_1^n x(1+x) + C_2^n x^2(1+x)^2 + \dots$

The coefficient of  $x^2 = C_1^n + C_2^n = \frac{2n + n^2 - n}{2} = 21$

$$n^2 + n - 42 = 0$$

$$(n-6)(n+7) = 0 \Rightarrow n = 6 \text{ or } n = -7 \text{ (Rejected)}$$

So  $n = 6$

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