

# Mathematics

# 數學科

## Functions

計數要小心，  
咪期望快一陣！





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### HKCEE Mathematics Syllabus – Functions:

Syllabus Topics	Notes (Whole Syllabus)	Notes (Foundation Part)	
6	Formulas. Functions and graphs.	Numerical applications. Change of subject, excluding formulas involving radicals. Simple algebraic fractions. Notation for function. Transformation on function. Graphs of $f(x) = ax + b$ and $f(x) = ax^2 + bx + c$ .	Numerical applications. Change of subject, excluding formulas involving radicals and algebraic fractions. Notation for function. Graphs of $f(x) = ax + b$ and $f(x) = ax^2 + bx + c$ .
		Knowledge of the general features of quadratic functions such as vertex, axis of symmetry and intercepts is required. The method of completing the square. Solving $f(x) > k$ , $f(x) < k$ , $f(x) \geq k$ and $f(x) \leq k$ graphically.	Knowledge of the general features of quadratic functions such as vertex, axis of symmetry and intercepts is required. Solving $f(x) > k$ , $f(x) < k$ , $f(x) \geq k$ and $f(x) \leq k$ graphically.
8	Exponential and logarithmic functions.	Graphs of exponential and logarithmic functions. Properties of logarithms, excluding the change of base. Applications of logarithm in real-life problems.	(This topic is not included.)

### 會考數學課程 - 函數

課題	註釋 (整體課程)	註釋 (基礎部分)	
6	公式。 函數及圖像。	數值計算之應用。主項之變換，不包括含有根號的公式。簡單代數分式。 函數之記號。函數之變換。 $f(x) = ax + b$ 之圖像及 $f(x) = ax^2 + bx + c$ 之圖像。 考生須認識二次函數之一般特性如頂點、對稱軸及截距。配方法。 利用圖解法求 $f(x) > k$ 、 $f(x) < k$ 、 $f(x) \geq k$ 及 $f(x) \leq k$ 的解。	數值計算之應用。主項之變換，不包括含有根號的公式。簡單代數分式。 函數之記號。 $f(x) = ax + b$ 之圖像及 $f(x) = ax^2 + bx + c$ 之圖像。 考生須認識二次函數之一般特性如頂點、對稱軸及截距。 利用圖解法求 $f(x) > k$ 、 $f(x) < k$ 、 $f(x) \geq k$ 及 $f(x) \leq k$ 的解。
8	指數函數及對數函數。	指數函數之圖像及對數函數之圖像。 對數性質，不包括換底公式。 對數在現實生活問題上之應用。	(不包括此課題)



**(A) Change of subject in formula 公式主項之變換**

e.g. Change the subject in the formula below to a.

<p>a. <math>e = c + a^3b</math></p>	<p>b. <math>d = \frac{a^2b + ef}{ac}</math></p>	<p>c. <math>e = ab\sqrt{c + f - \frac{a}{d}}</math></p>
<p>黎 Sir 提提你  :</p> $e - c = a^3b$ $\frac{e - c}{b} = a^3$ $\sqrt[3]{\frac{e - c}{b}} = a$ $a = \sqrt[3]{\frac{e - c}{b}}$	<p>黎 Sir 提提你  :</p> $acd = a^2b + ef$ $a^2b - acd = -ef$ $a^2b - acd + ef = 0$ $a = \frac{cd \pm \sqrt{(cd)^2 - 4bef}}{2b}$	<p>黎 Sir 提提你  :</p> <p><b>Will not be examined in HKCEE Maths</b></p>

**(B) Notation and transformation on function**

函數之記號和變換

What is Function 函數?

黎 Sir 提提你 :

1. Function is a machine, which will have different output [  $f(x)$  or  $y$  ] when having different input [  $x$  ]

2. Function Transformation:  $y = \pm c \bullet f(ax \pm b) \pm d$

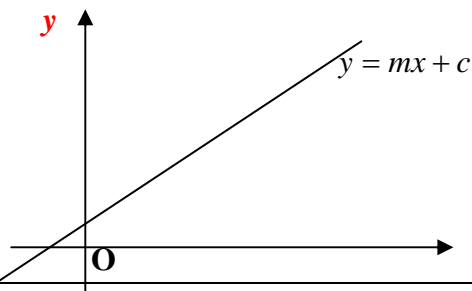


## (C) Knowledge and sketching of linear and quadratic

### function 線性和二次函數的知識和繪圖

黎 Sir 提提你  :

**Linear Function:**  $f(x) = mx + c$  or  $y = mx + c$ , with slope  $m$  and y-intercept  $c$ .



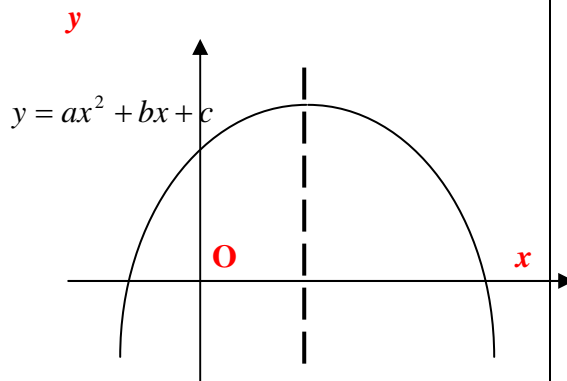
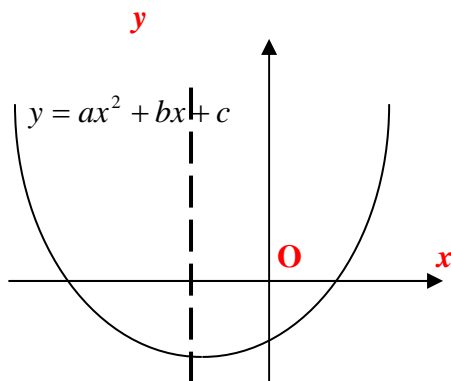
$Ax + By + C = 0$  into  $y = mx + c$

slope is  $-\frac{A}{B}$  and y-intercept is  $-\frac{C}{B}$ .



黎 Sir 提提你  :

**Quadratic Function:**  $f(x) = ax^2 + bx + c$



- Symmetry 對稱:**  
A parabola is symmetrical about this vertical line.
- Vertex 頂點:**
  - The vertex of a parabola is the point where the line of symmetry cuts the parabola.
  - At the vertex, the parabola is either at its minimum point 最低點 or maximum point 最高點.
- Direction of opening 開口方向:**
  - $a > 0$ , opening upwards 向上開口  
The vertex is the minimum point 最低點 of the parabola.
  - $a < 0$ , opening downwards 向下開口  
The vertex is the maximum point 最高點 of the parabola.
- y-intercept y 軸截距:**
  - The parabola cuts the y-axis at a point.
  - The value of y when  $x=0$ .
- x-intercept x 軸截距:**
  - The parabola cuts the x-axis at a point
  - May be 2 distinct points, 1 point or no point.
  - The value of x when  $y=0$

**Questions: But How to find Vertex? 怎樣找頂點?**

**Answers: Method of Completing Square. 配方法.**



## Method of Completing Square 配方法

Usage: To find the vertex 找頂點

$$y = ax^2 + bx + c \Rightarrow y = a(x - h)^2 + k$$

$$y = ax^2 + bx + c$$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$y = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b^2}{4a^2}\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

The vertex  $(h, k)$  of the quadratic function  $y = ax^2 + bx + c$  are as follows:

$$h = -\frac{b}{2a}, \quad k = -\frac{b^2 - 4ac}{4a}$$

e.g. find the vertex of the following functions.

$$y = x^2 - 6x - 5$$

$$y = [x^2 - 2x(3) + 3^2] - 3^2 - 5$$

$$y = (x - 3)^2 - 14$$

**Vertex: (3, -14), Minimum Point**

$$y = -3x^2 + 15x - 8$$

$$y = -3(x^2 - 5x) - 8$$

$$y = -3(x^2 - 2x(2.5) + 2.5^2) - 8 + (3)(2.5^2)$$

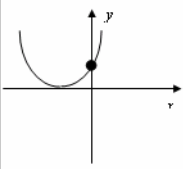
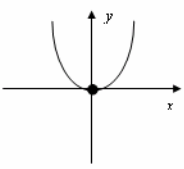
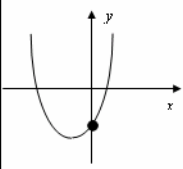
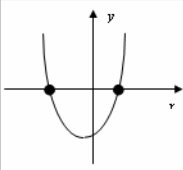
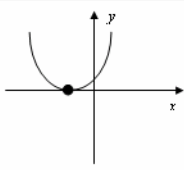
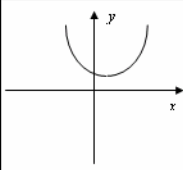
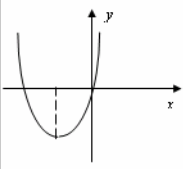
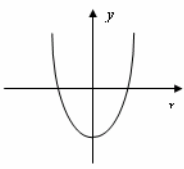
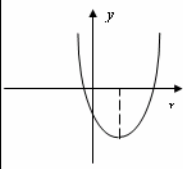
$$y = -3(x - 2.5) - 8 + (3)(2.5^2)$$

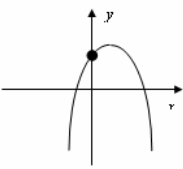
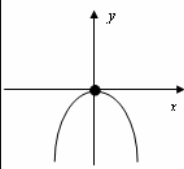
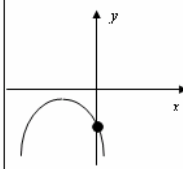
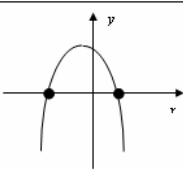
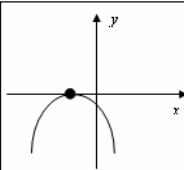
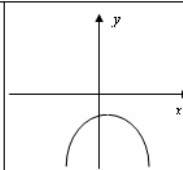
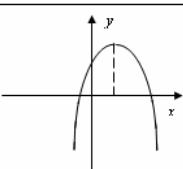
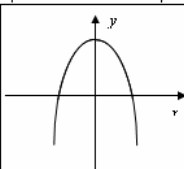
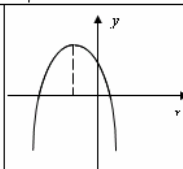
$$y = -3(x - 2.5) + 10.75$$

**Vertex: (2.5, 10.75), Maximum Point**



Summary of the Quadratic functions 二次函數總結

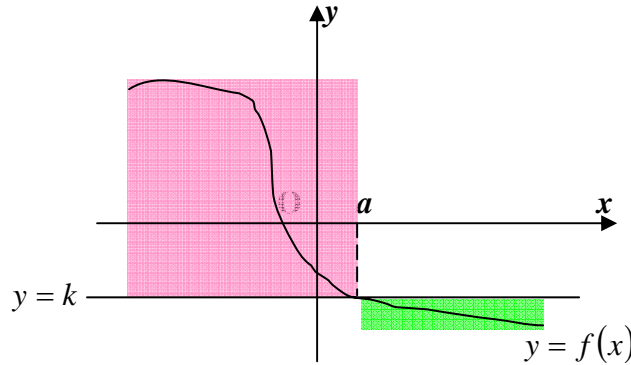
If $a > 0$		
$c$ is $y$ -intercept		
		
$c > 0$	$c = 0$	$c < 0$
$\Delta$ is number of $x$ -intercepts		
		
$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$b$ is position of line of symmetry		
		
$b > 0$	$b = 0$	$b < 0$

If $a < 0$		
$c$ is $y$ -intercept		
		
$c > 0$	$c = 0$	$c < 0$
$\Delta$ is number of $x$ -intercepts		
		
$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$b$ is position of line of symmetry		
		
$b > 0$	$b = 0$	$b < 0$



## Solving Inequalities by Graphical Method

### 以圖解法解不等式



i. $f(x) > k, x < a$	ii. $f(x) < k, x > a$	iii. $f(x) \geq k, x \leq a$	iii. $f(x) \leq k, x \geq a$
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e.g.  $x^2 - 6x - 5 > 0$

黎 Sir 提提你 :

e.g.  $x^2 - 6x - 5 < 0$

黎 Sir 提提你 :

e.g.  $x^2 - 6x - 5 < -14$

黎 Sir 提提你 :

e.g.  $x^2 - 6x - 5 > -14$

黎 Sir 提提你 :



(D) Exponential and logarithmic functions 指數和對數函數

**Exponential Function:**  $y = ka^x$ ,  $a > 0$ ,  $a \neq 1$ ,  $k \neq 0$ .

e.g.  $y = 4(5^x)$ ,  $-2\left(\frac{1}{3}\right)^x$

e.g. Sketch the curve of  $f(x) = 2^x$

黎 Sir 提提你  :

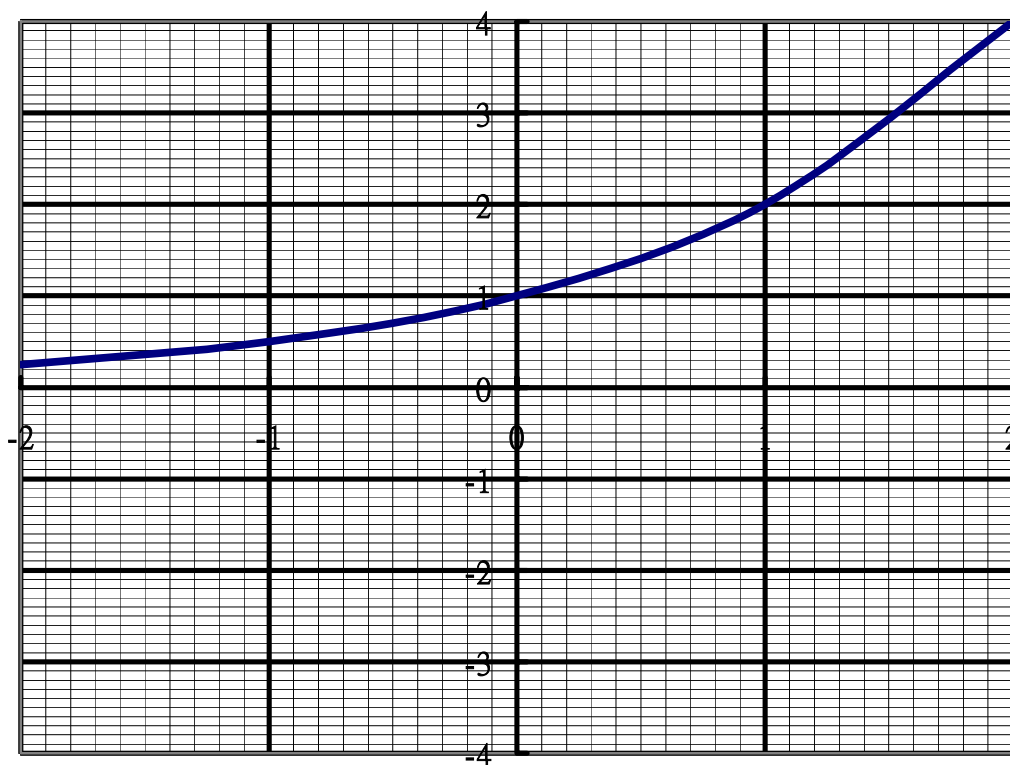


### Properties of Exponential Functions:

a.  $k > 0$  and  $a > 1$  e.g.  $f(x) = 2^x$

黎 Sir 提提你 :

$x$	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

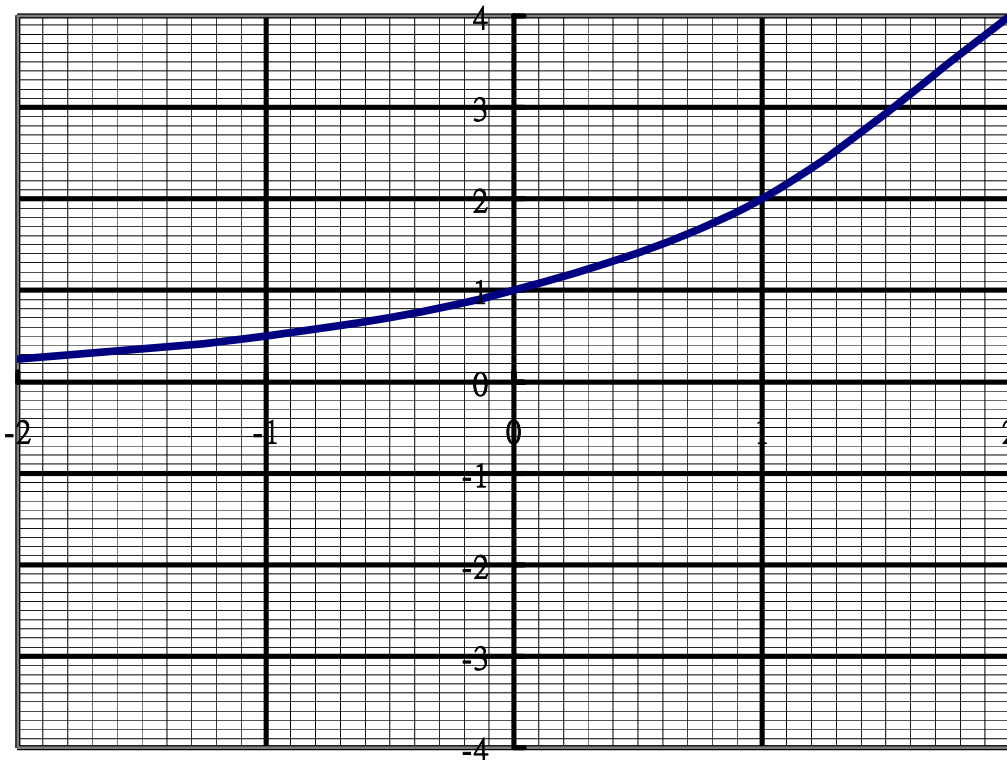




b.  $k < 0$  and  $a > 1$  e.g.  $f(x) = -2^x$

黎 Sir 提提你  :

$x$	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = -2^x$					

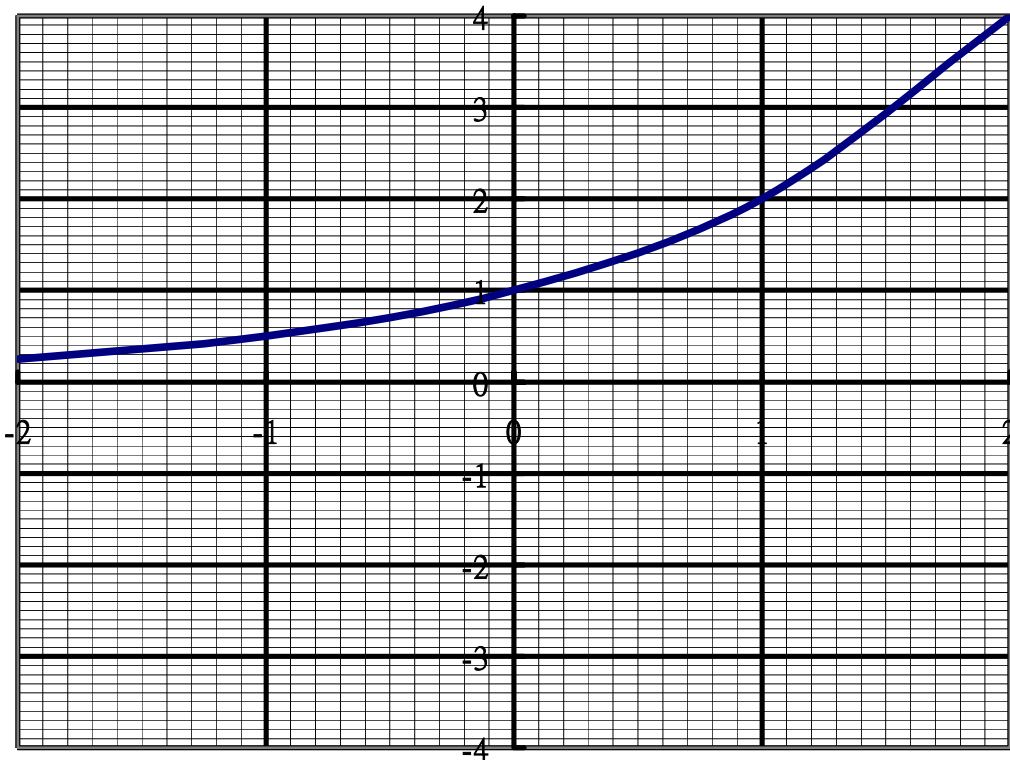




c.  $k > 0$  and  $0 < a < 1$  e.g.  $f(x) = \left(\frac{1}{2}\right)^x$

黎 Sir 提提你 :

$x$	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \left(\frac{1}{2}\right)^x$					

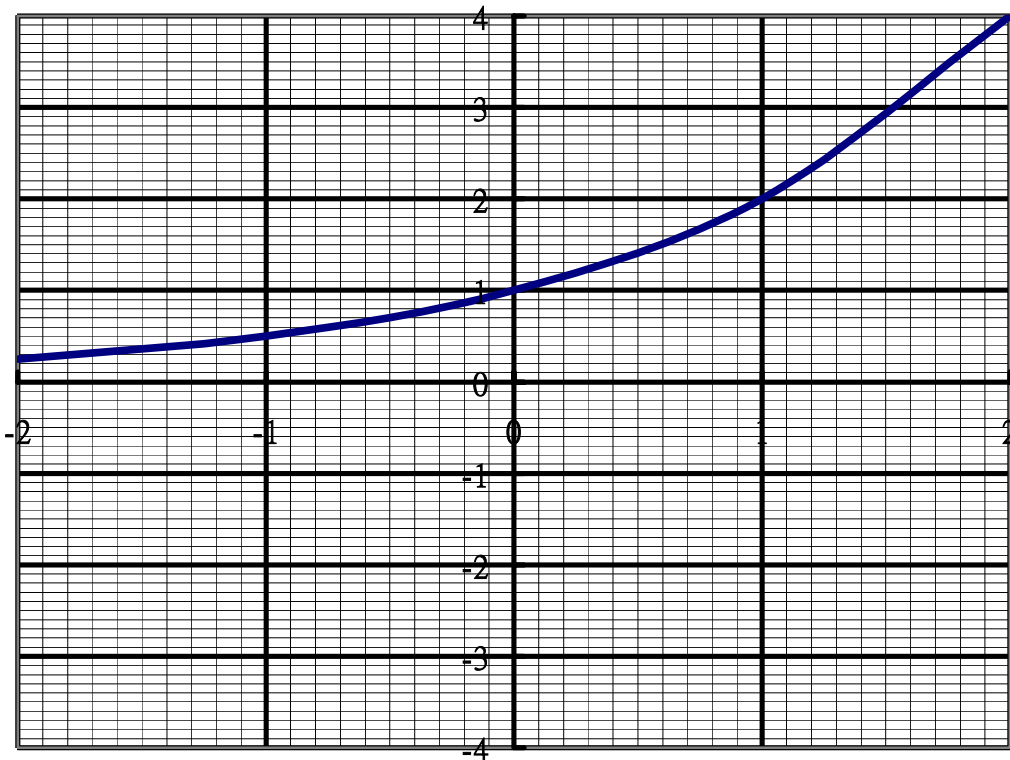




d.  $k < 0$  and  $0 < a < 1$  e.g.  $f(x) = -\left(\frac{1}{2}\right)^x$

黎 Sir 提提你  :

$x$	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = -\left(\frac{1}{2}\right)^x$					





**Application Exponential Problems: Depreciation**

e.g. The value of a car decreases 2% each year. Its value is \$150000 this year.

- (a) What will be the value of the car after  $t$  years?

Express your answer in terms of  $t$ .

- (b) What will be the value of the car after 10 years?

Correct your answer to the nearest dollar.

黎 Sir 提提你  :

- (a) The value of the car after  $t$  years

$$= 150000(1 - 2\%)^t$$

$$= 150000(98\%)^t$$

- (b) The value of the car after 10 years

$$= 150000(1 - 2\%)^{10}$$

$$= 150000(98\%)^{10}$$

$$= \$122,561$$



**Application Exponential Problems: Populations**

e.g. The population of a certain type of cell is doubled to the previous year.

Assume the population of that cell is 10 today.

(a) Complete the table.

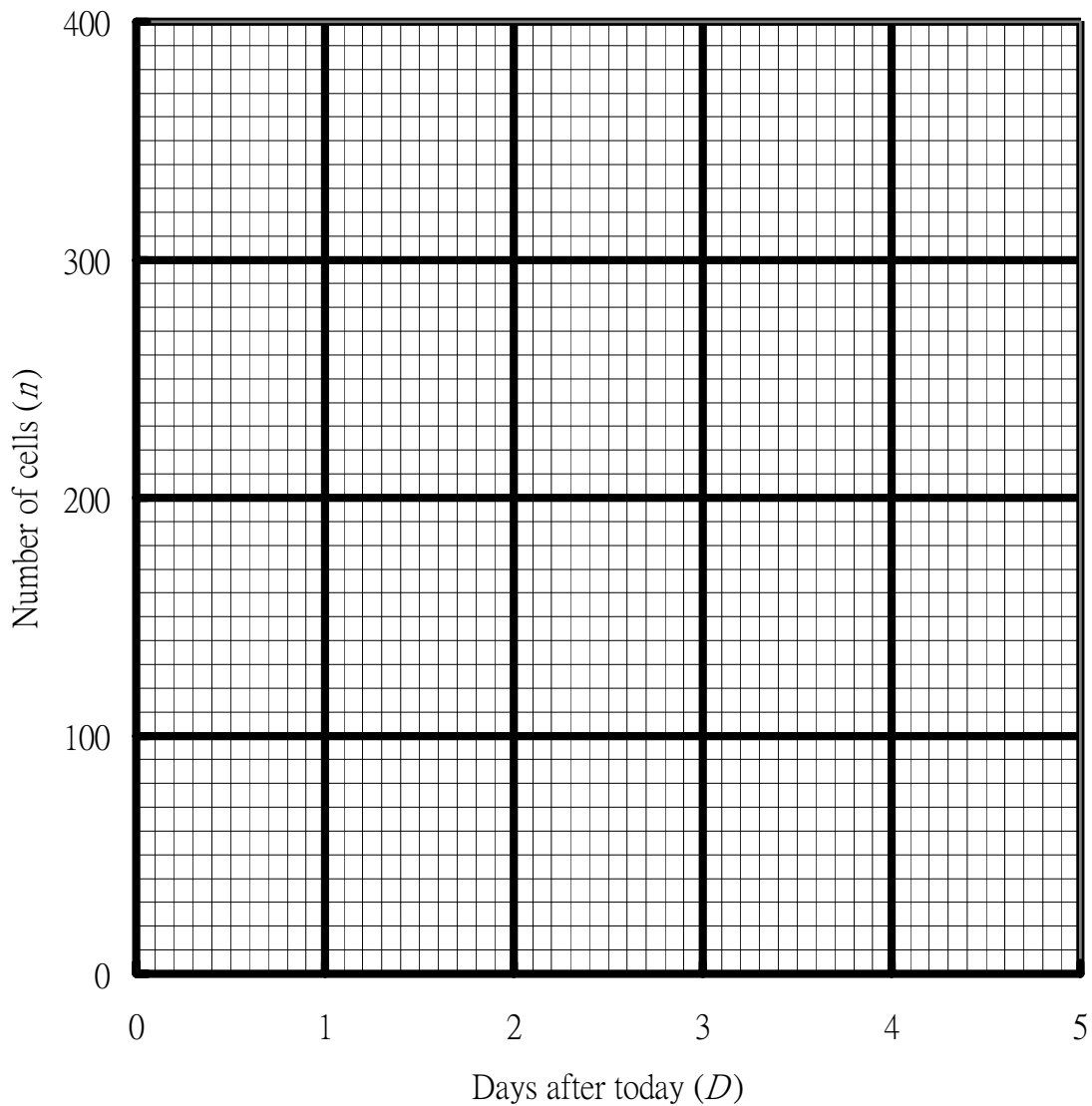
Days after today ( $D$ )	0	1	2	3	4	5
Number of cells ( $n$ )	10	20	40	80	160	320

(b) Express  $n$  in terms of  $D$ .

(c) Find  $n$  when  $D = 15$ .

(d) Plot the graph  $n$  against  $D$ .

(e) Estimate the minimum number of days required for 200 cells to be produced.





黎 Sir 提提你  :

(a) Refer to the table above

(b)  $n = 10(2^D)$

(c)  $n = 10(2^{15}) = 327680$   
 $= 327680$

(d) 5

$$200 = 10(2^D)$$

$$20 = (2^D)$$

$$\log 20 = \log 2^D$$

$$\log 20 = D \log 2$$

$$D = \frac{\log 20}{\log 2} = 4.32 \approx 5$$



**Logarithmic Functions:**  $y = \log x$ ,  $x > 0$ .

e.g.  $y = \log x$

黎 Sir 提提你  :

1. Definition of Log 對數定義

If 若  $y = 10^x$ , Then 則  $x = \log_{10} y$  or 或  $x = \log y$

2.  $\log 10 = 1$ ,  $\log 1 = 0$ ,  $\log 0 = \text{undefined}$  未定義,

4.  $\log(-k) = \text{undefined}$  未定義,  $k$  is any positive number 任意正數

5.  $\log(MN) = \log M + \log N$ ,  $M, N > 0$

6.  $\log \frac{M}{N} = \log M - \log N$ ,  $M, N > 0$

7.  $\log M^n = n \log M$ ,  $M > 0$

8.  $10^{\log a} = a$

**Application Logarithmic Problems: The Decibel Scale**

Decibel (dB) is the unit for measuring the loudness ( $L$ ) of sound, which is defined

as  $L = 10 \log \frac{I}{I_0}$ , where  $I$  is the intensity of sound and  $I_0$  is the threshold of

hearing. As scientists have found that  $I_0 = 10^{-12} \text{ W/m}^2$ .

e.g. Given that the intensity of a conversation is  $0.001 \text{ W/m}^2$ , find the loudness of the noise in decibels.

黎 Sir 提提你 :

$$\begin{aligned} L &= 10 \log \frac{I}{I_0} \\ &= 10 \log \frac{0.001}{10^{-12}} \\ &= 10 \log(10^9) \\ &= 10(9) \\ &= 90 \text{ dB} \end{aligned}$$

e.g. The noise level of a Mong Kok MTR station is 50 dB normally. When a MTR train enters the platform, the sound intensity record. is 100 times the noise level of the MTR station at normal. Find the loudness of sound in decibels of the sound intensity when the MTR train enters the platform.

黎 Sir 提提你 :

$$\begin{aligned} L_n &= 10 \log \frac{I_n}{I_0} = 50, \\ L &= 10 \log \frac{100I_n}{I_0} \\ &= 10(\log 100 + \log \frac{I_n}{I_0}) \\ &= 10 \log 100 + 10 \log \frac{I_n}{I_0} \\ &= 100 + 50 \\ &= 150 \text{ dB} \end{aligned}$$



### Application Logarithmic Problems: Richter scale ( $M$ )

Richter scale ( $M$ ): A Scale for measuring Earth Quake

$$\log E = 4.8 + 1.5M$$

e.g. The magnitude of Taiwan's 921 earthquake was 7.3 in the Richter scale. Find the energy released from the earthquake.

黎 Sir 提提你  :

$$\log E = 4.8 + 1.5M$$

$$\log E = 4.8 + 1.5(7.3)$$

$$\log E = 15.75$$

$$E = 10^{15.75}$$

$$E = 5.6 \times 10^{15} \text{ J}$$



## Exam Types Questions:

e.g.  $f(x) = \frac{x}{1-x}$ ,  $f\left(\frac{1}{x}\right) = ?$

黎 Sir 提提你 :

$$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$= \frac{\frac{1}{x}}{\frac{x-1}{x}}$$

$$= \frac{1}{x} \times \frac{x}{x-1}$$

$$= \frac{1}{x-1}$$

**Remarks: Chang in variable in functions**

e.g. If  $f(x) = \frac{1}{2x+1}$  and  $g(x) = \frac{1}{3-2x}$ , what is the value of  $x$  such that

$$f(x) = g(x)?$$

黎 Sir 提提你 :

$$f(x) = g(x)$$

$$\frac{1}{2x+1} = \frac{1}{3-2x}$$

$$3-2x = 2x+1$$

$$4x = 2 \Rightarrow x = \frac{1}{2}$$

**Remarks: find variable in functions**



e.g. If  $f(x) = 2x + 1$  and  $g(x) = 2x - 1$ , then  $f[g(x)] = ?$

黎 Sir 提提你  :

$$\begin{aligned}f[g(x)] &= f[2x - 1] \\ &= 2(2x - 1) + 1 \\ &= 4x^2 - 2\end{aligned}$$

**Remarks: Composite functions**

e.g. If  $f(2x) = 4x^2 + 2x$ , then  $f(3) = ?$

黎 Sir 提提你  :

Let  $2x = 3$ ,

$$x = \frac{3}{2}$$

$$\begin{aligned}\therefore f(3) &= 4\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) \\ &= 9 + 3 = 12 \\ &= 12\end{aligned}$$

**Remarks: Change in variable in functions**

e.g. If  $7^x = 11$ , then  $x = ?$

黎 Sir 提提你  :

$$\begin{aligned}7^x &= 11 \\ \log 7^x &= \log 11 \\ x \log 7 &= \log 11 \\ x &= \frac{\log 11}{\log 7} \approx \mathbf{1.23}.\end{aligned}$$

**Remarks:  $\log a^b = b \log a$  !!!**



e.g. If  $\log x < 0$ , then what is the range of  $x$ ?

黎 Sir 提提你  :

$$\log x < 0$$

$$10^{\log x} < 10^0$$

$$x < 1$$

**Remarks: Log Properties**

e.g. If  $\log x^3 = a$ , then  $\log \sqrt{x} =$

黎 Sir 提提你  :

$$\log x^3 = a$$

$$3\log x = a$$

$$\log x = \frac{a}{3}$$

$$x = 10^{\frac{a}{3}}$$

$$\sqrt{x} = \sqrt{10^{\frac{a}{3}}}$$

$$\log \sqrt{x} = \log 10^{\frac{a}{6}}$$

$$\log \sqrt{x} = \frac{a}{6} \log 10$$

$$\log \sqrt{x} = \frac{a}{6} (1)$$

$$\log \sqrt{x} = \frac{a}{6}$$

**Remarks: Remarks: Log Properties**



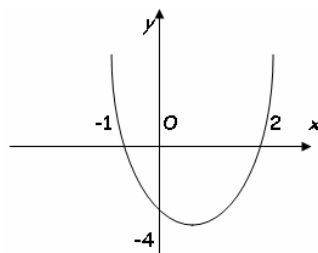
e.g. Given that  $\log 2 = a$  and  $\log 3 = b$ , then  $\log 15 =$

黎 Sir 提提你 :

$$\begin{aligned} \log 15 &= \log(3 \times 5) \\ &= \log 3 + \log 5 \\ &= b + \log \frac{10}{2} \\ &= b + \log 10 - \log 2 \\ &= b + 1 - a \\ &= b - a + 1 \end{aligned}$$

**Remarks: Remarks: Log Properties**

e.g. What is the quadratic function represents the curve?



黎 Sir 提提你 :

**Method 1 (Slow Method, Little Knowledge):**

Let  $f(x) = ax^2 + bx + c$

put **(-1,0), (2,0) and (0,-4)** give

$$0 = a + b + c,$$

$$0 = 4a + 2b + c,$$

$$-4 = c$$

**Solving the above 3 three equations give  $a = 2$ ,  $b = -2$  and  $c = -4$**

**Method 2 (Fast Method, More Knowledge):**

$$y = a(x - 2)(x + 1) \quad \{\text{Think: } [a(x - 2)(x + 1) = 0]\}$$

$$-4 = a(0 - 2)(0 + 1)$$

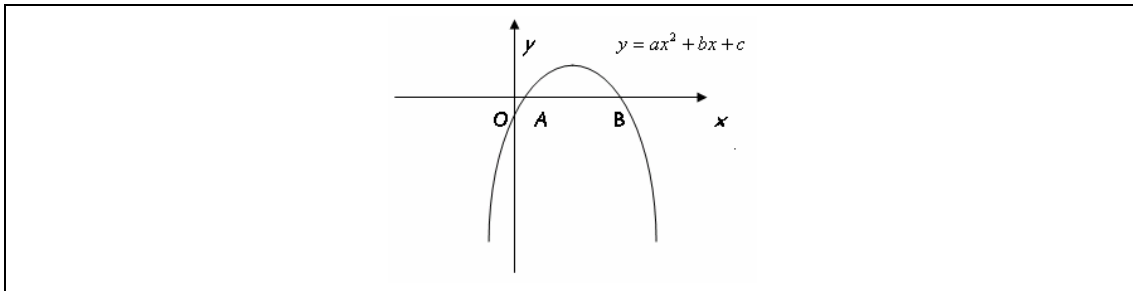
$$a = 2$$

$$y = 2(x - 2)(x + 1)$$

$$y = 2x^2 - 2x - 4$$



e.g. What is the range for a, b and c and  $\Delta$  of the following quadratic function?

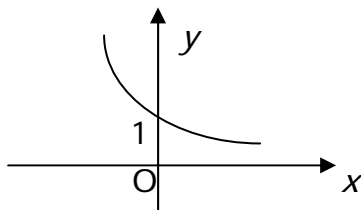


黎 Sir 提提你 :

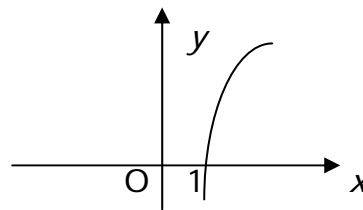
1. Opening Downwards:  $a < 0$ ,
2. Symmetry on +ve x-axis and  $a < 0$ :  $b > 0$
3. -ve y-intercept  $\Rightarrow c < 0$
4. distinct real roots  $\Rightarrow \Delta > 0$

e.g. Which of the following figures is the graph of  $y = 2^x$ ?

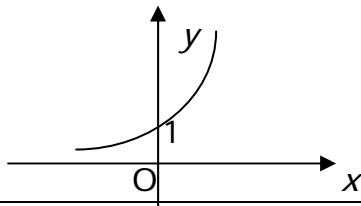
A.



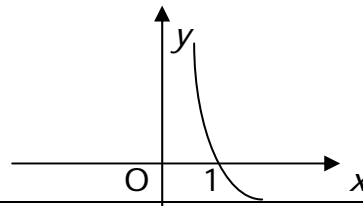
B.



C.



D.



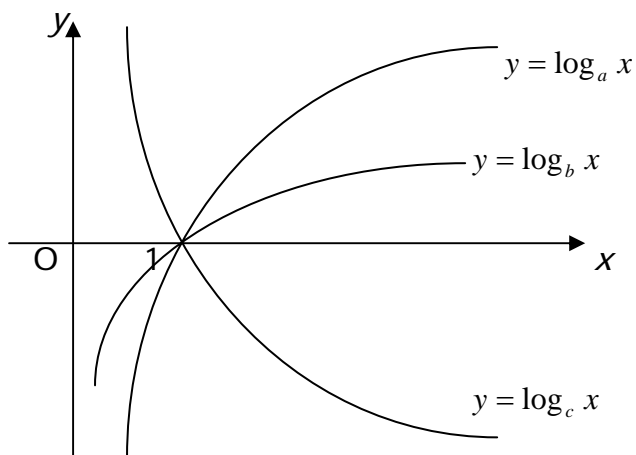
黎 Sir 提提你 : Answer: C

$x = 0, y = 2^0 = 1$ ,  
 when x increase,  $y = 2^x$  increases.  
 Answer: C



e.g. The following figure shows the sketch of the graphs of  $y = \log_a x$ ,

$y = \log_b x$  and  $y = \log_c x$ .



Which of the following is correct?

- A.  $c < a < b$     B.  $b < a < c$     C.  $a < b < c$     D.  $a < c < b$

黎 Sir 提提你  :



- e.g. (a) Describe the way of translating the graph of  $y = f(x)$  to the graph of  $y = f(x+1) + 2$ .
- (b) If  $f(x) = x^2$ , sketch the graphs of the functions  $y = f(x)$  and  $y = f(x+1) + 2$  on the same figure.

黎 Sir 提提你  :

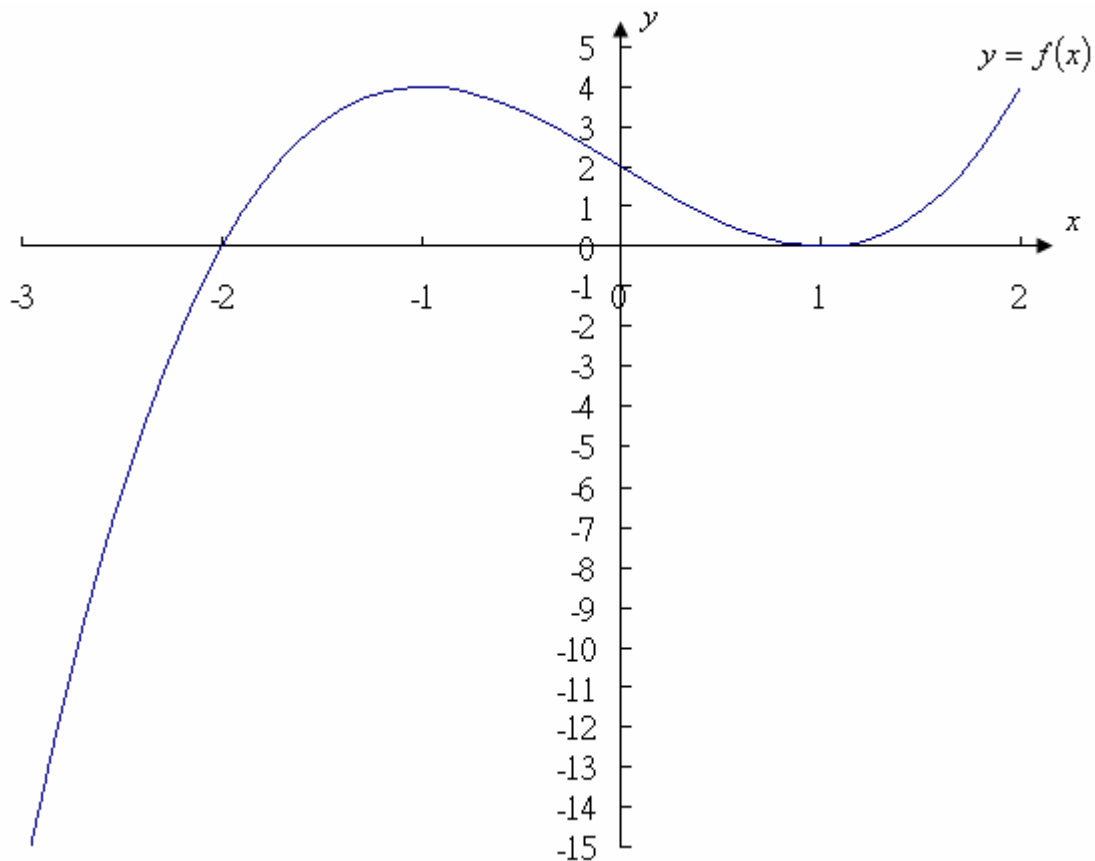


e.g. (a) Sketch the graph  $C_2 : y = f(-x)$  and  $C_3 : y = -f(x)$  on the same figure below for  $y = f(x)$ .

(b) Find the  $x$ -intercept and  $y$ -intercept of

(i)  $C_2 : y = f(-x)$

(ii)  $C_3 : y = -f(x)$



黎 Sir 提提你  :



Appendix I: Transformation on Functions 函數之變換

黎 Sir 提提你 :

$y = \pm c \cdot f(\pm ax \pm b) \pm d$

上/下移動 ->  $\pm d$

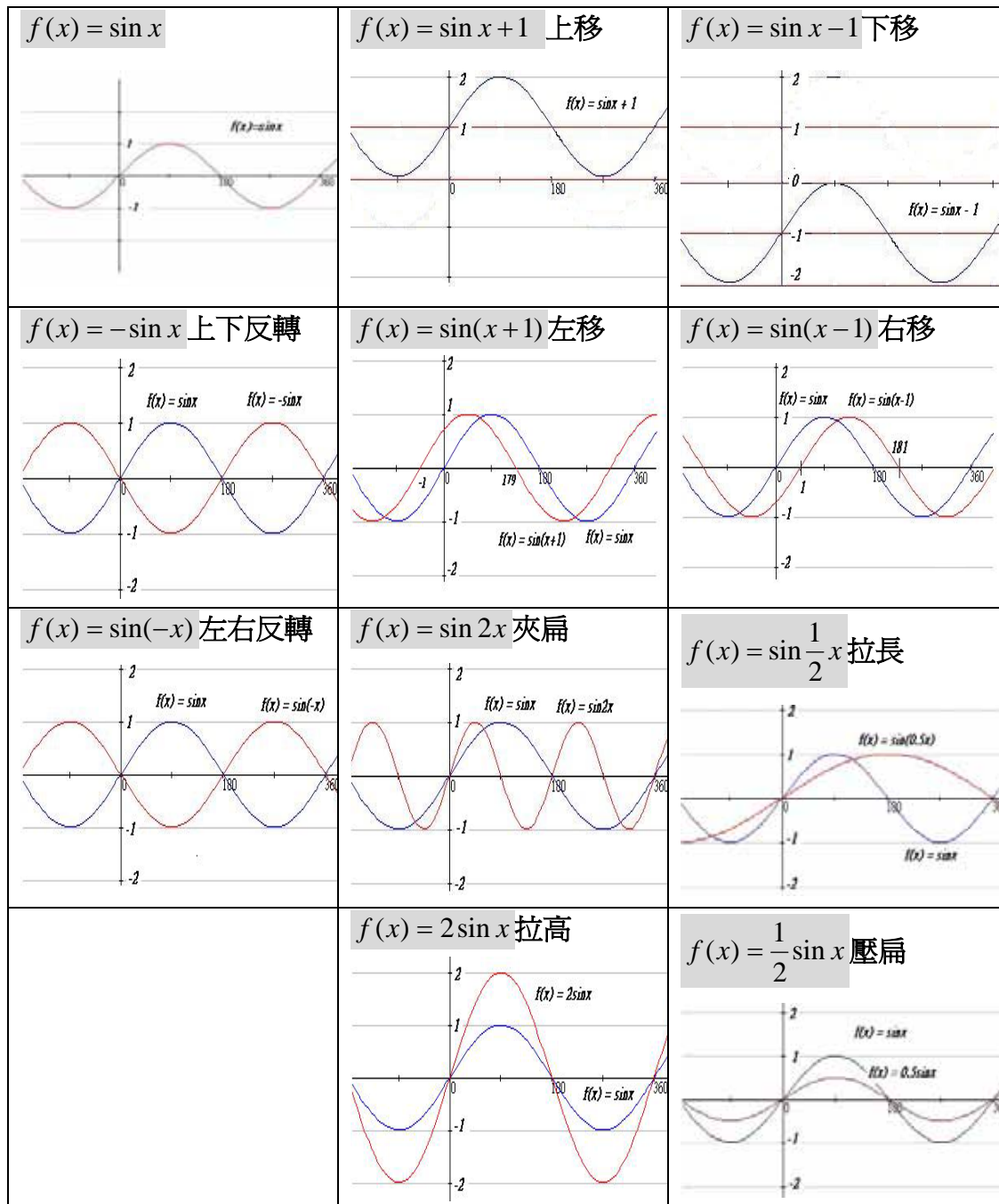
左/右移動 ->  $\pm b$

拉長/夾扁 ->  $0 < +a < 1, +a > 1$

左右反轉 ->  $-a$

拉高/壓扁 ->  $+c > 1, 0 < +c < 1$

上下反轉 ->  $-c$





Appendix II: Solving Inequalities by Graphical Method

以圖解法解不等式

<p>Graph of a parabola opening upwards. The x-axis has points <math>(1,0)</math> and <math>(2,0)</math> marked. The y-axis has a point <math>(1.5, -0.25)</math> marked. A vertical dashed line represents the axis of symmetry at <math>x=1.5</math>. Labels include "axis of symmetry 對稱軸" and "min. value of y 的極小值".</p>	<p>e.g Solve x for <math>x^2 - 3x + 2 &gt; 0</math></p> <p>黎 Sir 提提你  :</p> $x^2 - 3x + 2 > 0$ $(x-1)(x-2) > 0$ $x < 1 \text{ or } x > 2$
<p>Graph of a parabola opening upwards. The x-axis has points <math>(1,0)</math> and <math>(2,0)</math> marked. The y-axis has a point <math>(1.5, -0.25)</math> marked. A vertical dashed line represents the axis of symmetry at <math>x=1.5</math>. Labels include "axis of symmetry 對稱軸" and "min. value of y 的極小值".</p>	<p>e.g. Solve x for <math>x^2 - 3x + 2 &lt; 0</math></p> <p>黎 Sir 提提你  :</p> $x^2 - 3x + 2 < 0$ $(x-1)(x-2) < 0$ $1 < x < 2$
<p>Graph of a parabola opening upwards. The x-axis has points <math>(1,0)</math> and <math>(2,0)</math> marked. The y-axis has a point <math>(1.5, -0.25)</math> marked. A vertical dashed line represents the axis of symmetry at <math>x=1.5</math>. Labels include "axis of symmetry 對稱軸" and "min. value of y 的極小值".</p>	<p>e.g. Solve x for <math>x^2 - 3x + 2 &lt; -0.25</math></p> <p>黎 Sir 提提你  :</p> $x^2 - 3x + 2 < -0.25$ <p><b>There is no solution in x 無解.</b></p>
<p>Graph of a parabola opening upwards. The x-axis has points <math>(1,0)</math> and <math>(2,0)</math> marked. The y-axis has a point <math>(1.5, -0.25)</math> marked. A vertical dashed line represents the axis of symmetry at <math>x=1.5</math>. Labels include "axis of symmetry 對稱軸" and "min. value of y 的極小值".</p>	<p>e.g. Solve x for <math>x^2 - 6x - 5 &gt; -0.25</math></p> <p>黎 Sir 提提你  :</p> $x^2 - 3x + 2 > -0.25$ <p><b>All real values of x except 1.5</b> <b>所有 x 的實數值除了 1.5</b></p>

The End.